Algebra Univers. (2025) 86:11

1420-8911/25/020001-16
published online April 3, 2025
https://doi.org/10.1007/s00012-025-00889-5

Algebra Universalis

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Monadic ortholattices: completions and duality

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Abstract. We show that the variety of monadic ortholattices is closed under MacNeille and canonical completions. In each case, the completion of L is obtained by forming an associated dual space X that is a monadic orthoframe. This is a set with an orthogonality relation and an additional binary relation satisfying certain conditions. For the MacNeille completion, X is formed from the non-zero elements of L, and for the canonical completion, X is formed from the proper filters of L. The corresponding completion of L is then obtained as the ortholattice of bi-orthogonally closed subsets of X with an additional operation defined through the binary relation of X. With the introduction of a suitable topology on an orthoframe, as was done by Goldblatt and Bimbó, we obtain a dual adjunction between the categories of monadic ortholattices and monadic orthospaces. A restriction of this dual adjunction provides a dual equivalence.

Mathematics Subject Classification. 06C15, 06B23 06E15.

Keywords. Monadic ortholattice, MacNeille completion, Canonical completion, Duality, Orthoframe, Orthogonality space.

1. Introduction

Monadic algebras were introduced by Halmos [8] as an algebraic realization of the one-variable fragment of first-order logic. A monadic algebra is a Boolean algebra with an additional unary operation $\exists$, called a quantifier, whose closed elements are a Boolean subalgebra. Halmos' polyadic algebras [8] had a family of interrelated quantifiers and played the same role for full first-order logic. At about the same time, Henkin, Monk, and Tarski [10,11] introduced the closely related cylindric algebras as algebraic models of first-order logic. These too had a family of quantifiers, related in a somewhat different way than in

Presented by P. Jipsen.

polyadic algebras. They showed that each monadic algebra and each cylindric algebra can be embedded into a complete, atomic one. These were among the results that grew into the theory of canonical extensions of Boolean algebras with operators [12,13]. The approach was to show that the quantifier of a monadic algebra yields an equivalence relation on its set of ultrafilters, and the powerset of this relational structure is then a complete atomic monadic algebra extending the original. Including the Stone topology into this process yields a duality between monadic algebras and Stone spaces equipped with a compatible equivalence relation.

An *ortholattice* (abbrev.: OL) is a bounded lattice with an order-inverting period two complementation. A *monadic* OL is an ortholattice with a quantifier, a closure operation whose closed elements are a sub-OL. Janowitz [14] first considered quantifiers on orthomodular lattices, and Harding [9] studied them, and cylindric OLs, for their connections to von Neumann algebras, in particular, to subfactors. The broad purpose of this note is to conduct a study for monadic OLs similar to that described for monadic algebras. We use a number of tools for this purpose.

An *orthoframe* (abbrev.: OF) is a set X with an *orthogonality relation*, a binary relation $\perp$ that is irreflexive and symmetric. Orthoframes are also commonly called orthosets and orthogonality spaces. Orthogonality relations are special examples of the polarities described by Birkhoff [2]. It is known that the bi-orthogonally closed sets of an OF form a complete OL. There are two well-used ways to construct an orthoframe from an OL: with X the set of non-zero elements of L, which we call *MacLaren's* OF (see [15]), and with X the set of proper filters of L, which we call *Goldblatt's* OF (see [7]).

In [7], Goldblatt introduced a topology on what we call the Goldblatt frame of an OL L. This has all sets $h(a) = \{x : a \in x\}$ for $a \in L$, and their set-theoretic complements, as a sub-basis. Goldblatt showed that this yields a Stone topology, and that the clopen bi-orthogonally closed sets of the Goldblatt frame form an OL that is isomorphic to L. Bimbó [1] introduced *orthospaces* (abbrev.: OS) as certain OFs with a Stone topology and order. She defined morphisms between OSs and thought to have produced a duality between the category of ortholattices and their homomorphisms and the category of orthospaces and their morphisms. We show that what is produced in [1] is a dual adjunction and with an additional condition on OSs called *ortho-sobriety*, a dual equivalence is obtained [4].

Harding [9] defined *monadic orthoframes* to be OFs with an additional binary relation satisfying certain conditions. He showed that the bi-orthogonally closed elements of a monadic OF form a monadic OL, and that the MacLaren OF of a monadic OL can be turned into a monadic OF whose bi-orthogonally closed elements contain the original monadic OL as a subalgebra.

In the second section of this note we provide preliminaries. In the third section we show that for a monadic OL L, the bi-orthogonally closed sets of the monadic OF constructed in [9] give the MacNeille completion of L in the sense of [6]. Thus, the variety of monadic OLs is closed under MacNeille completions, hence by [6] it is also closed under canonical completions. We next provide a similar description of the canonical extension of L via monadic OFs. As a first step, we show that the bi-orthogonally closed sets of the Goldblatt OF of an OL is its canonical completion. Then we construct from a monadic OL L a monadic OF structure on its Goldblatt OF and show that the bi-orthogonally closed subsets of this monadic OF yield the canonical extension of L.

In the fourth section we consider orthospaces. We provide an example to show that there is not a dual equivalence between OL's and OS's, and use the remainder of the section to show that there is a dual adjunction between these categories. This dual adjunction restricts to a dual equivalence with the additional condition of *ortho-sobriety* on an OS as was pointed out by Dmitrieva in [4].

In the final section, we adapt this adjunction to the setting of monadic orthoframes and call the resulting structures *monadic orthospaces*. We then show that there is a dual adjunction between the categories of monadic OLs and monadic OS's and that this provides a dual equivalence when restricted to the full sub-category of monadic OSs consisting of ortho-sober monadic OSs.

2. Preliminaries

Definition 2.1. An *ortholattice* $(L, \wedge, \vee, ', 0, 1)$ is a bounded lattice with an order-inverting period two complementation. A *monadic* OL is an OL with a quantifier $\exists$, i.e. a closure operator where the orthocomplement of a closed element is closed.

For an OL $(L, \wedge, \vee, ', 0, 1)$ we use L to denote both the OL and its underlying set since this will not cause confusion. We let L^* be the set of non-zero elements of L and $\mathcal{F}(L)$ be the set of proper, non-empty filters of L ordered by set inclusion. We use letters such as $a, b, c, etc.$ for elements of L and $x, y, z, etc.$ for elements of $\mathcal{F}(L)$.

Definition 2.2. Let L be an OL. For $a, b \in L^*$ set $a \perp b$ iff $a \leq b'$, and for $x, y \in \mathcal{F}(L)$ set $x \perp y$ iff there is $a \in L^*$ with $a \in x$ and $a' \in y$.

It is obvious that both relations are irreflexive and symmetric.

Definition 2.3. Call $(L^*, \perp)$ the *MacLaren* OF of L and $(\mathcal{F}(L), \perp)$ the *Goldblatt* OF of L.

For an OF $(X, \perp)$ we use X to denote both the OF and its underlying set since this will not cause confusion. For $S \subseteq X$ its *orthogonal* is $S^\perp = \{y \in X : x \perp y \text{ for all } x \in S\}$, and its *bi-orthogonal* is $S^{\perp\perp}$. Call S *bi-orthogonally closed* if $S = S^{\perp\perp}$.

Definition 2.4. Let $\mathcal{B}(X)$ be the set of bi-orthogonally closed subsets of an OF X.

It is well-known that this is a complete OL with partial ordering of set inclusion and with the orthocomplement of S given by $S^\perp$. In this OL meets are given by intersections, joins by the bi-orthogonal of the union, and the bounds are the emptyset and X.

Proposition 2.5. *Suppose L is an* OL. *Then there is an* OL *embedding $g : L \to \mathcal{B}(L^*, \perp)$ with $g(a) = \{b \in L^* : b \leq a\}$, and an* OL*-embedding $h : L \to \mathcal{B}(\mathcal{F}(L), \perp)$ with $h(a) = \{x : a \in x\}$.*

It is known [15] that $g : L \to \mathcal{B}(L^*, \perp)$ is the MacNeille completion of L. In the next section, we show that $h : L \to \mathcal{B}(\mathcal{F}(L), \perp)$ is the canonical extension of L [5]. In the following, for a binary relation R on a set X and $A \subseteq X$, we denote the relational image of A by $R[A] = \{y \in X : xRy$ for some $x \in A\}$.

Definition 2.6. A *monadic orthoframe* is a triple $(X, \perp, R)$ where $(X, \perp)$ is an OF and R is a reflexive, transitive binary relation on X that satisfies $R[R[\{x\}]^\perp] \subseteq R[\{x\}]^\perp$ for all $x \in X$.

Thus, a monadic orthoframe is an OF with a pre-order R where the orthogonal $R[\{x\}]^\perp$ of a principal pre-order ideal is a pre-order ideal. Partial orders on OFs were considered in [3]. The following results were established by Harding in [9].

Proposition 2.7. *For X a monadic* OF, *its bi-orthogonally closed subsets $\mathcal{B}(X)$ form a monadic* OL *under the quantifier $\exists A = R[A]^{\perp\perp}$.*

Proposition 2.8. *For L a monadic* OL, *the relation R on L^* defined by aRb iff $b \leq \exists a$ makes $(L^*, \perp, R)$ a monadic* OF, *and the map $g : L \to \mathcal{B}(L^*, \perp, R)$ is a monadic* OL *embedding.*

3. MacNeille and canonical completions

For L a bounded lattice, an n-ary operation $f : L^n \to L$ is called *monotone* if in each coordinate it either preserves or reverses order. Implication of a Heyting algebra is monotone, and both the quantifier and orthocomplementation of a monadic OL are monotone. A lattice with additional operations is monotone if each of its operations is monotone, and a variety of lattices with additional operations is monotone if each of its members is monotone. There is a theory of completions of lattices with monotone operations that we describe in the restricted case of the variety of monadic OLs.

For a bounded lattice L, its *MacNeille completion* is a pair $(e, \overline{L})$ where $\overline{L}$ is a complete lattice, $e : L \to \overline{L}$ is a lattice embedding, and each element of $\overline{L}$ is both a join and a meet of elements of the image of L. For a monadic OL L, its MacNeille completion is the bounded lattice $\overline{L}$ with unary operations $\overline{'}$ and $\overline{\exists}$ defined by

$$x^{\overline{'}} = \bigwedge\{e(a') : e(a) \leq x\}$$

$$\overline{\exists}x = \bigvee\{e(\exists a) : e(a) \leq x\}$$

Proposition 3.1. *For L a monadic* OL, *$g : L \to \mathcal{B}(L^*, \perp, R)$ is its MacNeille completion.*

Proof. On the OL level this is well known [15]. It remains to show that for $A \subseteq L^*$ bi-orthogonally closed, i.e. for a normal ideal A of L^*, we have

$$R[A]^{\perp\perp} = \bigvee\{R[g(a)]^{\perp\perp} : a \in A\}.$$

Since the join of bi-orthogonally closed sets is given by the closure of their union, the right side of this expression is equal to $(\bigcup\{R[g(a)]^{\perp\perp} : a \in A\})^{\perp\perp}$. By general principles, this in turn is equal to $(\bigcup\{R[g(a)] : a \in A\})^{\perp\perp}$, hence to $R[\bigcup\{g(a) : a \in A\}]^{\perp\perp}$. But $g(a)$ is the principal ideal generated by a and A is a normal ideal, so $\bigcup\{g(a) : a \in A\} = A$. □

Definition 3.2. For a bounded lattice L, its *canonical completion* is a pair (e, C) where C is a complete lattice and $e : L \to C$ is a bounded lattice embedding that is dense and compact. *Dense* means that each element of C is both a join of meets and a meet of joins of elements of the image of L. *Compact* means that if $S, T \subseteq L$ then

$$\bigwedge e[S] \leq \bigvee e[T] \quad \Rightarrow \quad \bigwedge e[S'] \leq \bigvee e[T']$$

for some finite $S' \subseteq S$ and $T' \subseteq T$.

Each lattice has up to isomorphism a unique canonical completion, and we call this *the* canonical completion, denoted by L^σ. An element of the canonical completion that is a meet of elements of the image of L is called closed, and the set of closed elements is $\mathcal{K}$. For a bounded lattice with additional monotone operations, there are extensions of the operations to the canonical completion. We describe these for orthocomplementation and a quantifier, where we call the extensions ${}'^\sigma$ and $\exists^\sigma$, by

$$x^{'^\sigma} = \bigwedge\left\{\bigvee\{e(a') : k \leq e(a)\} : k \leq x \text{ and } k \in \mathcal{K}\right\},$$
$$\exists^\sigma x = \bigvee\left\{\bigwedge\{e(\exists a) : k \leq e(a)\} : k \leq x \text{ and } k \in \mathcal{K}\right\}.$$

Proposition 3.3. *For L an* OL, *$h : L \to \mathcal{B}(\mathcal{F}(L), \perp)$ is its canonical extension.*

Proof. We recall a construction of the canonical completion of a bounded lattice L given in [5]. Let $\mathcal{I}_0$ and $\mathcal{F}_0$ be its sets of non-empty ideals and non-empty filters, respectively, and define a relation R_0 from $\mathcal{F}_0$ to $\mathcal{I}_0$ by $x\, R_0\, u \Leftrightarrow x \cap u \neq \emptyset$. For $A \subseteq \mathcal{F}_0$ and $B \subseteq \mathcal{I}_0$ set

$$\Phi_0(A) = \{u : x\, R_0\, u \text{ for all } x \in A\},$$
$$\Psi_0(B) = \{x : x\, R_0\, u \text{ for all } u \in B\}.$$

Then for $\mathcal{G}_0 = \{A : A = \Psi_0\Phi_0(A)\}$ the collection of Galois closed sets of this polarity, we have that $\mathcal{G}_0$ is a complete lattice under set inclusion and $h_0 : L \to \mathcal{G}_0$ given by $h_0(a) = \{x \in \mathcal{F}_0 : a \in x\}$ is a lattice embedding with $(h_0, \mathcal{G}_0)$ the canonical completion of L.

It is a simple matter to verify that if we repeat this construction using the sets $\mathcal{F}$ and $\mathcal{I}$ of proper non-empty filters and ideals, with relation $x\, R\, u \Leftrightarrow x \cap u \neq \emptyset$, corresponding maps Φ and Ψ, and resulting complete lattice $\mathcal{G} = \{A : A = \Psi\Phi(A)\}$, then there is an isomorphism $\alpha : \mathcal{G}_0 \to \mathcal{G}$ taking A to the set $A\backslash\{L\}$ obtained by removing the unique improper filter L from A. It follows that the map $h : L \to \mathcal{G}$ with $h(a) = \{x \in \mathcal{F} : a \in x\}$ is a lattice embedding and $(h, \mathcal{G})$ is a canonical completion of L.

We turn now to the case at hand where L is an OL. Note that the set $\mathcal{F}$ of proper non-empty filters of L is exactly the set $\mathcal{F}(L)$. Also, for a filter y of L let $y' = \{a' : a \in x\}$. Note that y' is a proper non-empty ideal of L and each such arises from a unique proper, non-empty filter y of L. Further, for the relation $\perp$ on $\mathcal{F}(L)$, we have $x \perp y \Leftrightarrow x\, R\, y'$. It follows that $\mathcal{G}$ is equal to the collection $\mathcal{B}(\mathcal{F}(L), \perp)$ of biorthogonally closed sets of $(\mathcal{F}(L), \perp)$ and $h : L \to \mathcal{B}(\mathcal{F}(L), \perp)$ is the canonical completion of L as a bounded lattice.

It remains to see that the orthocomplementation of $\mathcal{B}$ is the extension $'^{\sigma}$ described above. One notes that by monotonicity arguments, $h(a') = h(a)'^{\sigma}$ for each $a \in L$. The result follows using the general De Morgan laws and density of the canonical extension. □

Remark 3.4. It appears that Goldblatt [7, p. 47] claims that if L is a complete OL, then h maps L isomorphically onto $\mathcal{B}(\mathcal{F}(L), \perp)$. This is not the case, the canonical extension of a complete Boolean algebra is not usually an isomorphism.

We now consider Goldblatt's OF in the context of a monadic OL.

Proposition 3.5. *Let L be a monadic* OL *and define a binary relation R on its Goldblatt* OF *by $x\, R\, y$ iff $\exists[x] \subseteq y$. Then $X = (\mathcal{F}(L), \perp, R)$ is a monadic* OF *and $h : L \to \mathcal{B}(X)$ is the canonical extension of L.*

Proof. To show that X is a monadic OF we must first show that R is reflexive and transitive. For reflexivity, let $a \in \exists[x]$ so that $a = \exists b$ for some $b \in x$. Since $b \leq \exists b$ and x is upward closed, we have $\exists b = a \in x$ and therefore xRx. For transitivity, assume xRy and yRz so that $\exists[x] \subseteq y$ and $\exists[y] \subseteq z$. Let $a \in \exists[x]$ so that $a = \exists b$ for some $b \in x$. Then $\exists b \in y$ and hence $\exists\exists b \in \exists[y]$ which implies $\exists\exists b \in z$. Then $\exists\exists b = \exists b$ and $a = \exists b$ so $a \in z$. Therefore $\exists[x] \subseteq z$ and hence xRz so we conclude R is transitive.

We now show that R satisfies $R[R[\{x\}]^{\perp}] \subseteq R[\{x\}]^{\perp}$ for all $x \in \mathcal{F}(L)$. Let z be the filter generated by $\exists[x]$ and note that z is the smallest filter belonging to $R[\{x\}]$. Thus, $R[\{x\}]^{\perp} = \{z\}^{\perp}$. If $y \in \{z\}^{\perp}$, then there is $b \in z$ with $b' \in y$. Since z is the filter generated by $\exists[x]$, there are $a_1, \ldots, a_n \in x$ with $\exists a_1 \wedge \cdots \wedge \exists a_n \leq b$. Set $a = a_1 \wedge \cdots \wedge a_n$. Then $a \in x$ and we have $\exists a \leq \exists a_1 \wedge \cdots \wedge \exists a_n \leq b$. Thus $b' \leq (\exists a)'$ so $(\exists a)'$ belongs to y. It follows that

$$R[\{x\}]^{\perp} = \{z\}^{\perp} = \{y : (\exists a)' \in y \text{ for some } a \in x\}.$$

Suppose $y \in R[\{x\}]^{\perp}$ and $y\, R\, w$. Then there is $a \in x$ with $(\exists a)' \in y$ and $\exists[y] \subseteq w$. Since L is a monadic OL, we have $(\exists a)' = \exists(\exists a)' \in w$, giving $w \in R[\{x\}]^{\perp}$ as required. So X is a monadic OF.

For $a \in L$ we have that $h(a) = \{x : a \in x\}$. It follows that $R[h(a)] = \{x : \exists a \in x\}$. Indeed, if $x \in R[h(a)]$ then $y\, R\, x$ for some $y \in h(a)$. But then $a \in y$ and $\exists[y] \subseteq x$, so $\exists a \in x$. Conversely, if $\exists a \in x$, then $\uparrow a \in h(a)$ and $\exists[\uparrow a] \subseteq x$, so $x \in R[h(a)]$. It follows that $R[h(a)]$ is bi-orthogonally closed, so

$h(\exists a) = R[h(a)] = R[h(a)]^{\perp\perp} = \exists h(a)$. So, using $\mathcal{B}$ for $\mathcal{B}(X)$, we have that $h : L \to \mathcal{B}$ is a monadic OL-embedding. We further know that when restricted to the OL reduct, this is the canonical extension. It remains to show that the quantifier that we denote $\exists_R$ of $\mathcal{B}$ is the canonical extension $\exists^\sigma$ of quantifier of L, as described above.

Let $\mathcal{K}$ be the set of closed elements of $\mathcal{B}$, that is, those that are a meet of elements in the image of h, and for each filter x of L let $K_x = \{y : x \subseteq y\}$. Let $S \subseteq L$. If the filter x generated by S is proper, we have $\bigwedge h[S] = K_x$, and $\bigwedge h[S] = \emptyset$ if x is improper. Thus, each non-empty $K \in \mathcal{K}$ is of the form K_x for some proper filter x of L, and it is easily seen that this x is unique.

The definition of $\exists^\sigma$ gives

$$\exists^\sigma K_x = \bigwedge\{h(\exists a) : K_x \subseteq h(a)\}.$$

Since $K_x \subseteq h(a)$ iff $a \in x$, $\exists^\sigma K_x$ is equal to $\bigcap\{h(\exists a) : a \in x\}$, which in turn is equal to $\{y : \exists[x] \subseteq y\}$, and hence is given by $R[K_x]$. But this set is bi-orthogonally closed since it is the intersection of bi-orthogonally closed sets, so $\exists^\sigma K_x = R[K_x]^{\perp\perp} = \exists_R K_x$.

Suppose that A is any element of $\mathcal{B}$. Then, the definition of $\exists^\sigma$ and the result just established for closed elements K_x gives

$$\exists^\sigma A = \bigvee\{\exists^\sigma K_x : K_x \leq A\} = \bigvee\{\exists_R K_x : K_x \leq A\}.$$

Since $\exists_R$ is order preserving, $\exists^\sigma A \subseteq \exists_R A$. To see equality, suppose $y \in R[A]$. Then $x\,R\,y$ for some $x \in A$. Since A is bi-orthogonally closed, it is an upset in the poset of proper filters, so $K_x \subseteq A$. But $y \in R[K_x] = \exists_R K_x$. Thus $R[A] \subseteq \bigcup\{\exists_R K_x : K_x \leq A\}$. Using the established fact that $\exists^\sigma K_x = \exists_R K_x$ and taking the bi-orthogonal closure of both sides gives $\exists_R A \leq \bigvee\{\exists^\sigma K_x : K_x \leq A\} = \exists^\sigma A$. □

To conclude this section, we recall that in the classical setting, one associates to a monadic algebra $(B, \exists)$ a set X with an equivalence relation S on X. We show that this path can also be taken with a monadic OL.

Definition 3.6. For L a monadic OL, define a relation S on its set $\mathcal{F}(L)$ of proper filters by $x\,S\,y$ iff $\exists[x] = \exists[y]$.

Clearly S is an equivalence relation. To examine its properties further, we use an auxilliary relation $\uparrow$ on $\mathcal{F}(L)$ where $x \uparrow y$ iff $x \subseteq y$. We do this to emphasize that the domain of this particular instance of $\subseteq$ is the set $\mathcal{F}(L)$ and feel that this notation improves readability. With this, we then write $\uparrow\! S$ for the composite $\uparrow \circ\, S$. So $x \uparrow\! S\, z$ iff there is y with $\exists[x] = \exists[y]$ and $y \subseteq z$.

Lemma 3.7. $R = \uparrow\! S$.

Proof. Given x, let $\widehat{x}$ be the filter generated by $\exists[x]$ and note that since x is down-directed, $\widehat{x}$ is the upset generated by $\exists[x]$. Then $x\,R\,z$ iff $\exists[x] \subseteq z$ iff $\widehat{x} \subseteq z$. Since $\exists[x] = \exists[\widehat{x}]$ it follows that $x\,R\,z$ implies $x \uparrow\! S\, z$. Conversely, if $x \uparrow\! S\, z$, there is y with $\exists[x] = \exists[y]$ and $y \subseteq z$. Then $\exists[x] = \exists[y] \subseteq y \subseteq z$ and so $x\,R\,z$. □

Proposition 3.8. *Let L be a monadic* OL*. Then $Y = (\mathcal{F}(L), \perp, S)$ is a monadic* OF*, S is an equivalence relation, and $h : L \to \mathcal{B}(Y)$ is the canonical extension of L.*

Proof. Since $x \leq y$ and $x \perp z$ implies $y \perp z$, for any $A \subseteq Y$ we have $A^\perp = (\uparrow A)^\perp$. Thus $S[\{x\}]^\perp = (\uparrow S[\{x\}])^\perp = R[\{x\}]^\perp$. Then, since $S[A] \subseteq R[A]$ for any subset A, we have

$$S[S[\{x\}]^\perp] \subseteq R[R[\{x\}]^\perp] \subseteq R[\{x\}]^\perp = S[\{x\}]^\perp.$$

Here, the second containment uses the fact that $X = (\mathcal{F}(L), \perp, R)$ is a monadic OF. This inequality shows that Y is a monadic OF, and as noted, S is an equivalence relation. Since the mapping h does not depend on the choice of R or S, it is an OL embedding into $\mathcal{B}(Y)$ that provides a canonical extension of L when considered as a OL. To show that it is a canonical extension of L as a monadic OL, we show that the quantifiers $\exists_R$ and $\exists_S$ on $\mathcal{B}(\mathcal{F}(L), \perp)$ are equal. But

$$\exists_R A = R[A]^{\perp\perp} = (\uparrow S[A])^{\perp\perp} = S[A]^{\perp\perp} = \exists_S A.$$

This completes the proof. □

Our derivation could have been done throughout starting with the relation S, but it would have been a bit more complicated.

4. Orthospaces

Bimbó [1] placed Goldblatt's work [7] on the Stone space of a OL in a categorical setting in an attempt to create a duality between the category OL of OLs and their homomorphisms and what she called the category of orthospaces.

The focus of this section is to show that what is obtained in [1] is a dual adjunction that gives rise to a dual equivalence when the definition of orthospaces is appropriately extended.

Definition 4.1. An orthospace (abbrev.: OS) $(X, \perp, \leq, \tau)$ consists of an OF with a partial ordering $\leq$ and a compact topology τ that satisfies

(1) if $x \not\leq y$ then there is $U \in \mathcal{C}(X)$ with $x \in U$ and $y \notin U$,
(2) if $x \perp z$ and $x \leq y$, then $y \perp z$,
(3) if $U \in \mathcal{C}(X)$, then $U^\perp \in \mathcal{C}(X)$,
(4) if $x \perp y$, then there is $U \in \mathcal{C}(X)$ with $x \in U$ and $y \in U^\perp$.

Here $\mathcal{C}(X)$ is the set of clopen, bi-orthogonally closed subsets of X.

Condition (1) guarantees that every OS X is totally-order disconnected and therefore totally disconnected. Since X is compact by definition, X is a Stone space.

Lemma 4.2. *In any* OS *we have $x \leq y$ iff $y \in \{x\}^{\perp\perp}$.*

Proof. If $x \leq y$ then $y \in \{x\}^{\perp\perp}$ since (2) implies that each bi-orthogonally closed set is an upset and we always have $x \in \{x\}^{\perp\perp}$. Conversely, if $x \not\leq y$, then by (1) there is $U \in \mathcal{C}(X)$ with $x \in U$ and $y \notin U$. Since U is bi-orthogonally closed and $x \in U$, we have $\{x\}^{\perp\perp} \subseteq U$, hence $y \notin \{x\}^{\perp\perp}$. □

Using this lemma, one can formulate an equivalent definition of an OS that does not involve an ordering. Call $(X, \perp, \tau)$ an OS$'$ if $(X, \perp)$ is an OF with a compact topology τ that satisfies (3) and (4) and additionally has $\mathcal{C}(X)$ separate points, that is, it satisfies the following: (1$'$) if $x \neq y$ there is $U \in \mathcal{C}(X)$ with $U \cap \{x, y\}$ containing one element. Clearly if $(X, \perp, \leq, \tau)$ is an OS, then $(X, \perp, \tau)$ is an OS$'$ since (1) implies (1$'$).

Proposition 4.3. *If $(X, \perp, \tau)$ is an* OS$'$*, then setting $x \leq y$ iff $y \in \{x\}^{\perp\perp}$, we have that $(X, \perp, \leq, \tau)$ is an* OS*.*

Proof. It is simple to see that $\leq$ is reflexive and transitive. If $x \neq y$, then by (1$'$) there is U that separates them, say $x \in U$ and $y \notin U$. It follows that $\{x\}^{\perp\perp} \subseteq U$, hence $y \notin \{x\}^{\perp\perp}$. So $\leq$ is anti-symmetric, hence a partial order. If $x \not\leq y$, then $y \notin \{x\}^{\perp\perp}$. So there is z with $x \perp z$ and $y \not\perp z$. By (4) there is $U \in \mathcal{C}(X)$ with $x \in U$ and $z \in U^{\perp}$. We cannot have $y \in U$ since that would give $y \perp z$, hence $y \notin U$. This shows that (1) holds for our derived relation $\leq$. Suppose $x \perp z$ and $x \leq y$. Then $y \in \{x\}^{\perp\perp}$. But $z \in \{x\}^{\perp}$, so $y \perp z$, giving (2). □

If we begin with an OS$'$ $(X, \perp, \tau)$, then form an OS $(X, \perp, \leq, \tau)$ as above, then build from it an OS$'$, we obviously return to the original since we have merely created and then discarded an auxiliary relation $\leq$. Suppose we start with an OS $(X, \perp, \leq, \tau)$, and then use the OS$'$ to form a partial ordering. By Lemma 4.2, we return to our original OS. Thus, the notions of OS and OS$'$ are equivalent. We follow Bimbó's terminology to make it easy to match with her paper, although the notion of an OS$'$ seems simpler.

Definition 4.4. Let $(P, \perp, \leq, \tau)$ and $(X, \perp, \leq, \tau)$ be OS's. A function $\phi : P \to X$ is an OS *morphism* if ϕ is continuous and satisfies

(1) if $\phi(p) \perp \phi(q)$ then $p \perp q$,
(2) if $x \not\perp \phi(p)$ then there exists q with $q \not\perp p$ and $\phi(q) \in \{x\}^{\perp\perp}$.

Let OS be the category of OS's and their morphisms.

For an OL L, Goldblatt [7] considered the topology τ on $\mathcal{F}(L)$ having as a sub-basis all sets $h(a)$, and their set-theoretic complements, for $a \in L$. He showed that this is a Stone topology, that the clopen bi-orthogonally closed sets of $\mathcal{F}(L)$ form an OL, and that h is an isomorphism from L to the OL of clopen bi-orthogonally closed sets of $\mathcal{F}(L)$. Bimbó showed [1, Lemma 3.4] that $(\mathcal{F}(L), \subseteq, \perp, \tau)$ is an OS, and that for any OS X, its clopen bi-orthogonally closed sets $\mathcal{C}(X)$ form an OL [1, Lemma 3.3]. So for any OL L we have an OS $\mathcal{F}(L)$, and for any OS X we have an OL $\mathcal{C}(X)$.

Proposition 4.5. *These assignments on objects extend to contravariant functors $\mathcal{F} : \mathsf{OL} \to \mathsf{OS}$ and $\mathcal{C} : \mathsf{OS} \to \mathsf{OL}$. For $f : L \to M$ an* OL*-homomorphism, $\mathcal{F}(f) : \mathcal{F}(M) \to \mathcal{F}(L)$ is given by $\mathcal{F}(f) = f^{-1}[\cdot]$ and for $\phi : P \to X$ an* OS*-morphism, $\mathcal{C}(\phi) : \mathcal{C}(X) \to \mathcal{C}(P)$ is given by $\mathcal{C}(\phi) = \phi^{-1}[\cdot]$.*

Proof. See [1, Lemmas 3.9 and 3.10] for the proofs. □

It was shown in [1] that there are a pair of natural transformations $h : 1_{\mathsf{OL}} \to \mathcal{CF}$ and $g : 1_{\mathsf{OS}} \to \mathcal{FC}$ where, for an OL L and OS X, the components of the natural transformations are given by:

$$h_L : L \to \mathcal{CF}(L) \text{ is given by } h_L(a) = \{x : a \in x\},$$
$$g_X : X \to \mathcal{FC}(X) \text{ is given by } g_X(x) = \{U : x \in U\}.$$

The naturality of g and h is given in [1, Thm. 3.11]. The proof that h_L is an OL-isomorphism is given in [7]. Thus $h : 1_{\mathsf{OL}} \to \mathcal{CF}$ is a natural isomorphism. In [1, Thm. 3.6], it is claimed that g_X is an isomorphism, hence that $\mathcal{F}$ and $\mathcal{C}$ provide a dual equivalence between OL and OS. The proof that g_X is one-one is correct, but it need not be onto. The issue with g_X being onto was first pointed out in [4]. Below, we provide an example to show that g_X need not be onto.

Example 4.6. For $X = \{x, y\}$, let $\perp$ be the relation $\neq$ of inequality, $\leq$ be the relation $=$ of equality, and τ be the discrete topology. Then τ is a Stone topology on X, $\perp$ is irreflexive and symmetric, hence an orthogonality relation on X, and $\leq$ is a partial ordering. Moreover, every subset of X is both clopen and bi-orthogonally closed, so $\mathcal{C}(X)$ is the powerset of X and $U^{\perp}$ is the set-theoretic complement of U for each $U \subseteq X$. It is a simple matter to verify that $(X, \perp, \leq, \tau)$ is an OS. Then $\mathcal{C}(X)$ is a 4-element Boolean algebra, since it is the powerset of the 2-element set X. But a 4-element Boolean algebra has 3 proper filters. So $\mathcal{FC}(X)$ is a 3-element OS, thus cannot be isomorphic to X.

Definition 4.7. An OS X is *ortho-sober* if each proper filter in the ortholattice $\mathcal{C}(X)$ is equal to $\{U \in \mathcal{C}(X) : x \in U\}$ for some $x \in X$.

Ortho-sober orthospaces were introduced by Dmitrieva [4], and later considered by McDonald and Yamamoto [16]. The point is that for the full subcategory OSOS of OS consisting of ortho-sober orthospaces, $\mathcal{F}$ maps OL into OSOS, and then $\mathcal{F}$ and $\mathcal{C}$ provide a dual equivalence between OL and OSOS. We use the remainder of this section to formulate what exists in the approach of [1] without the introduction of the ortho-sober condition.

Theorem 4.8. *The functors $\mathcal{F} \dashv \mathcal{C}$ provide an adjunction between OL and OS^{op}.*

Proof. For L an OL and X an OS we define mappings $(\,\cdot\,)^-$ and $(\,\cdot\,)^+$

$$\text{Hom}_{\mathsf{OL}}(L, \mathcal{C}(X)) \underset{(\,\cdot\,)^+}{\overset{(\,\cdot\,)^-}{\rightleftarrows}} \text{Hom}_{\mathsf{OS}}(X, \mathcal{F}(L))$$

by setting for $f : L \to \mathcal{C}(X)$ and $\phi : X \to \mathcal{F}(L)$

$$f^-(x) = \{a : x \in f(a)\},$$
$$\phi^+(a) = \{x : a \in \phi(x)\}.$$

Note that $f^- : X \to \mathcal{F}(L)$ is the composite of $g_X : X \to \mathcal{FC}(X)$ and $\mathcal{F}(f) : \mathcal{FC}(X) \to \mathcal{F}(L)$. Indeed, we have

$$\mathcal{F}(f) \circ g_X(x) = \mathcal{F}(f)(\{U : x \in U\}) = \{a : x \in f(a)\}.$$

Note also that $\phi^+ : L \to \mathcal{C}(X)$ is the composite of $h_L : L \to \mathcal{C}\mathcal{F}(L)$ and $\mathcal{C}(\phi) : \mathcal{C}\mathcal{F}(L) \to \mathcal{C}(X)$. Indeed, we have

$$\mathcal{C}(\phi) \circ h_L(a) = \mathcal{C}(\phi)(\{y : a \in y\}) = \{x : a \in \phi(x)\}.$$

Thus, the maps $(\,\cdot\,)^-$ and $(\,\cdot\,)^+$ between hom-sets are well-defined. Also, we have

$$f^{-+}(a) = \{x : a \in f^-(x)\} = \{x : x \in f(a)\} = f(a),$$
$$\phi^{+-}(x) = \{a : x \in \phi^+(a)\} = \{a : a \in \phi(x)\} = \phi(x).$$

Thus, for each L, X the maps $(\,\cdot\,)^-$ and $(\,\cdot\,)^+$ are mutually inverse bijections between hom-sets. We require naturality. For naturality in one coordinate, we must show that if $\psi : X' \to X$ and $\alpha : X \to \mathcal{F}(L)$, then $(\alpha \circ \psi)^+ = \mathcal{C}(\psi) \circ \alpha^+$. For naturality in the other coordinate, we must show that if $f : L' \to L$ and $\alpha : X \to \mathcal{F}(L)$, then $(\mathcal{F}(f) \circ \alpha)^+ = \alpha^+ \circ f$. For the former, we have

$$\begin{aligned}(\mathcal{C}(\psi) \circ \alpha^+)(a) &= \mathcal{C}(\psi)(\{x : a \in \alpha(x)\}) \\ &= \{x' : \psi(x') \in \{x : a \in \alpha(x)\}\} \\ &= \{x' : a \in \alpha\psi(x')\} \\ &= (\alpha \circ \psi)^+(a),\end{aligned}$$

and for the latter, we have

$$\begin{aligned}(\alpha^+ \circ f)(b) &= \{x : f(b) \in \alpha(x)\} \\ &= \{x : b \in (\mathcal{F}(f) \circ \alpha)(x)\} \\ &= (\mathcal{F}(f) \circ \alpha)^+(b).\end{aligned}$$

This completes the proof. □

General categorical considerations yield the following.

Corollary 4.9. *The category* OL *is equivalent to* OSOS^{op} *and* OSOS^{op} *is a co-reflective subcategory of* OS^{op}.

5. Monadic orthospaces

In this section, we extend the results in the previous section to the setting of monadic OLs.

Definition 5.1. A tuple $(X, \perp, \leq, R, \tau)$ is a monadic orthospace if $(X, \perp, \leq, \tau)$ is an OS, $(X, \perp, R)$ is a monadic OF, and for each $U \in \mathcal{C}(X)$ we have $R[U] \in \mathcal{C}(X)$.

For a monadic OF X, its bi-orthogonally closed sets $\mathcal{B}(X)$ form a monadic OL under the quantifier $\exists A = R[A]^{\perp\perp}$. It is clear from the definition of a monadic OS that its clopen bi-orthogonally closed sets $\mathcal{C}(X)$ form a subalgebra of $\mathcal{B}(X)$, hence form a monadic OL.

Definition 5.2. Let L be a monadic OL, and equip its Goldblatt OS $(\mathcal{F}(L), \perp, \leq, \tau)$ with the relation xRy iff $\exists[x] \subseteq y$ of its Goldblatt OF. Call this the monadic Goldblatt OS and denote it $\mathcal{F}L$.

For a monadic OL L, we have that $\mathcal{F}(L)$ is indeed a monadic OS. It is clearly an OS and also a monadic OF. It remains only to show that if U is a clopen and bi-orthogonally closed set of $\mathcal{F}(L)$, then so is $R[U]$. By Goldblatt's result, $U = h(a)$ for some $a \in L$. In the proof of Proposition 3.5, we saw that $R[h(a)] = h(\exists a)$, so $R[U]$ is clopen and bi-orthogonally closed, hence $\mathcal{F}(L)$ is a monadic OS.

Proposition 5.3. *For L a monadic* OL*, the map $h_L : L \to \mathcal{CF}(L)$ given by $h_L(a) = \{x : a \in x\}$ is a monadic* OL *isomorphism.*

Proof. We know that h_L is an OL isomorphism. By the discussion above, for $a \in L$ we have $h_L(\exists a) = R[h_L(a)] = R[h_L(a)]^{\perp\perp} = \exists h_L(a)$ and thus h_L is a homomorphism for $\exists$. □

Definition 5.4. For monadic OS's X and Y, a map $\phi : X \to Y$ is a monadic OS morphism if it is an OS morphism and $R[\phi^{-1}[U]] = \phi^{-1}[R[U]]$ for each $U \in \mathcal{C}(Y)$.

Proposition 5.5. *For X a monadic* OS*, the map $g_X : X \to \mathcal{FC}(X)$ given by $g_X(x) = \{U : x \in U\}$ is a monadic* OS *embedding.*

Proof. To aid readability, we write g for g_X. It is known that g is an OS embedding. The remaining condition for g to be a monadic OS morphism involves several levels. To assist with this, we use the following conventions. Elements of X are written x, y and R is the additional relation on X. Elements of $\mathcal{FC}(X)$ are filters of $\mathcal{C}(X)$ and are written as F. The relation of $\mathcal{FC}(X)$ is denoted S. We must show that for $\mathcal{V} \in \mathcal{CFC}(X)$

$$R[g^{-1}[\mathcal{V}]] = g^{-1}[S[\mathcal{V}]].$$

Note that by Goldblatt's result, there is some $U_0 \in \mathcal{C}(X)$ with

$$\mathcal{V} = \{F \in \mathcal{FC}(X) : U_0 \in F\}.$$

Observe that $x \in g^{-1}[\mathcal{V}]$ iff $g(x) \in \mathcal{V}$, which occurs iff $U_0 \in g(x)$, hence, iff $x \in U_0$. Thus

$$R[g^{-1}[\mathcal{V}]] \; = \; R[U_0].$$

For $y \in X$ we have $y \in g^{-1}[S[\mathcal{V}]]$ iff $g(y) \in S[\mathcal{V}]$. This occurs iff there is some $F \in \mathcal{V}$ with $F \, S \, g(y)$, hence some filter F with $U_0 \in F$ and $\exists_{\mathcal{C}(X)}[F] \subseteq g(y)$. But $\uparrow U_0$, the principal filter of $\mathcal{C}(X)$ generated by U_0, is the smallest filter containing U_0. So these conditions occur iff $\exists_{\mathcal{C}(X)} U_0 \in g(y)$, and this occurs iff $y \in \exists_{\mathcal{C}(X)} U_0$. But we have seen that $\exists_{\mathcal{C}(X)} U_0 = R[U_0]$. Thus

$$g^{-1}[S[\mathcal{V}]] \; = \; R[U_0].$$

This establishes the result. □

Lemma 5.6. *The composite of monadic* OS *morphisms is a monadic* OS *morphism.*

Proof. Suppose X, Y, Z are monadic OS's with associated relations R, S, T, respectively. Let $\phi : X \to Y$ and $\psi : Y \to Z$ be monadic OS morphisms. Then for $W \in \mathcal{C}(Z)$ we have that $\psi^{-1}[W] \in \mathcal{C}(Y)$ since ψ is in particular an OS morphism, and then $S[\psi^{-1}[W]]$ also belongs to $\mathcal{C}(Y)$ since Y is a monadic OS. Then $\phi^{-1}\psi^{-1}[T[W]] = \phi^{-1}[S[\psi^{-1}[W]]] = R[\phi^{-1}\psi^{-1}[W]]$. □

Let mOL be the category of monadic OLs (usually MOL denotes modular OLs) and mOS be the category of monadic OSs. For L a monadic OL, its Goldblatt frame $\mathcal{F}(L)$ is a monadic OS, and for a monadic OS X its clopen bi-orthogonally closed subsets $\mathcal{C}(X)$ form a monadic OL.

Lemma 5.7. *For $f : L \to M$ an mOL morphism and $\phi : X \to Y$ an mOS morphism, $f^{-1} : \mathcal{F}(M) \to \mathcal{F}(L)$ is an mOS morphism, and $\phi^{-1} : \mathcal{C}(Y) \to \mathcal{C}(X)$ is an mOL morphism.*

Proof. We first show that $\psi = f^{-1}$ is an mOS morphism. Since we already know it is an OS morphism, it remains to show that for $U \in \mathcal{CF}(L)$, we have $R[\psi^{-1}[U]] = \psi^{-1}[R[U]]$. By Goldblatt's result, $U = h_L(a)$ for some $a \in L$. Earlier we showed that $R[h_L(a)] = h_L(\exists a)$ and $\psi^{-1}[h_L(a)] = h_M(f(a))$. Thus, we have

$$\begin{aligned} R[\psi^{-1}[U]] &= R[h_M(f(a))] = h_M(\exists f(a)) = h_M(f(\exists a)) \\ &= \psi^{-1}[h_L(\exists a)] = \psi^{-1}[R[U]]. \end{aligned}$$

Since $\phi : X \to Y$ is an OS morphism, ϕ^{-1} is an OL homomorphism. We must show that for $U \in \mathcal{C}(Y)$, that $\phi^{-1}[\exists U] = \exists\phi^{-1}[U]$. But $U \in \mathcal{C}(Y)$ implies that $R[U] \in \mathcal{C}(Y)$, and so $\exists U = R[U]^{\perp\perp} = R[U]$. Therefore, since ϕ is a mOS morphism, we have

$$\phi^{-1}[\exists U] = \phi^{-1}[R[U]] = R[\phi^{-1}[U]] = \exists\phi^{-1}[U],$$

which completes the proof. □

We then have that $\mathcal{F}$ and $\mathcal{C}$ are contravariant functors between mOL and mOS. Recall, that for an OL L and OS X, we earlier produced mutually inverse bijections between homsets where $f^- = \mathcal{F}(f) \circ g_X$ and $\phi^+ = \mathcal{C}(\phi) \circ h_L$.

$$\mathrm{Hom}_{\mathsf{OL}}(L, \mathcal{C}(X)) \underset{(\cdot)^+}{\overset{(\cdot)^-}{\rightleftarrows}} \mathrm{Hom}_{\mathsf{OS}}(X, \mathcal{F}(L))$$

If L is an mOL, X an mOS, f is an mOL morphism, and ϕ an mOS morphism, then since g_X is an mOS morphism, and h_L is an mOL morphism, it follows that f^- is an mOS morphism and ϕ^+ is an mOL morphism. Thus, we have mutually inverse bijections

$$\mathrm{Hom}_{\mathsf{mOL}}(L, \mathcal{C}(X)) \underset{(\cdot)^+}{\overset{(\cdot)^-}{\rightleftarrows}} \mathrm{Hom}_{\mathsf{mOS}}(X, \mathcal{F}(L))$$

The naturality of these in each coordinate is given by the naturality in the previous setting. This yields the following.

Theorem 5.8. *$\mathcal{F} \dashv \mathcal{C}$ is an adjunction between* mOL *and* mOS^{op}*.*

Recall that an OS X is ortho-sober if each proper filter of $\mathcal{C}(X)$ is equal to $\{U : x \in U\}$ for some $x \in X$. This is equivalent to having $g_X : X \to \mathcal{FC}(X)$ be an isomorphism. The dual adjunction between OL and OS restricts to a dual equivalence between OL and the full subcategory of ortho-sober OSs.

Corollary 5.9. *There is a dual equivalence between* mOL *and the full subcategory* OSmOS *of* mOS *consisting of ortho-sober monadic* OS*s.*

Corollary 5.10. *The category* OSmOS^{op} *is a co-reflective subcategory of* mOS^{op}*.*

Acknowledgements

The authors wish to thank the referees for carefully reading the manuscript and providing valuable suggestions that improved the presentation of the paper.

Author contributions All authors contributed equally.

Funding The first and third listed authors were partially supported by US Army grant W911NF-21-1-0247 and the first author was also partially supported by NSF grant DMS-2231414. The second author was supported by CGS-D MSFSS grant no. 771-2023-0044 and CGS-D SSHRC grant no. 767-2022-1514.

Data availability Not applicable.

Declarations

Ethical approval Not applicable.

Conflict of interest Not applicable.

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Received: 20 December 2023.
Accepted: 13 January 2025.